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Finite Strain in Elastic Problems

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VIII—Finite Strain in Elastic Problems

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I—Introduction

1—*Components of Finite Strain*

The mathematical theory of Elasticity, as at present developed, is based on the assumption that the displacements that have to be considered in elastic solid bodies are so small that the squares and products of the first differential coefficients of u, v, w with respect to x, y, z can be neglected in comparison with their first powers. That is why we cannot use it to get a satisfactory solution of many a problem in elasticity in which the displacement is finite and the strain produced is not small enough to justify the above assumption. For example, we can bend any rectangular plate in the form of a cylinder by couples applied to the edges only. As far as I know there exists no exact solution for this simple problem. In Section III I have attempted to give its solution based on the theory of finite strain.

The first step towards the solution of the type of problems we have in mind is to get the components of strain corresponding with any displacement. Like the body-stress equations, these should be referred to the actual position of a point P of the material in the strained condition, and not to the position of a point considered before strain. The importance of this point, overlooked by various authors, cannot be exaggerated. Apparently Filon and Coker* were the first to notice it and stress its importance. To a first approximation it is immaterial which method of reference is adopted, for the values of the strain components in the two cases differ only in the second order terms. In the first case they are given by such expressions as

$$S_x = \frac{\partial u}{\partial x} - \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \quad \ldots \ldots \quad (1.1)$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right], \quad \ldots \ldots \quad (1.2)$$

* "Treatise on Photo-Elasticity," p. 188.

and in the second by

$$S_x = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \quad \ldots \quad (2.1)$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right]. \quad \ldots \quad (2.2)$$

For our present purpose we must use the values of S_x, σ_{yz}, etc., given by equations such as (1.1) and (1.2). We shall, therefore, call x, y, z the co-ordinates of the actual position of a point P in the strained solid.

An important point may be noticed here. The fact that two or more solutions are additive in the ordinary theory ceases to hold good after the introduction of the square terms in the values of S_x, σ_{yz}, etc.

2—*Conditions to be Satisfied by the Displacement*

It is obvious that without some restrictions placed on the displacement the mathematical solution of an elastic problem treated on the hypothesis of a finite displacement is not going to be unique. This is illustrated by the well-known example of a piece of rubber tubing turned inside out. We cannot now use the well-known Consistency equations given by

$$(1+\eta)\,\nabla^2\widehat{xx} + \frac{\partial^2}{\partial x^2}(\widehat{xx} + \widehat{yy} + \widehat{zz}) = 0,$$

$$(1+\eta)\,\nabla^2\widehat{yz} + \frac{\partial^2}{\partial y\,\partial z}(\widehat{xx} + \widehat{yy} + \widehat{zz}) = 0, \text{ etc.}$$

The new Consistency equations are obtained by eliminating u, v, w between the stress-strain relations. These equations will be very complex in character and cannot be of any important use to us. The conditions that u, v, w must always satisfy are that they should be both differentiable and continuous throughout the region occupied by the body. These are implicitly involved in obtaining the strain components given by (1) or (2).

3—*Form of the Boundary*

In all problems of the type under consideration we must know the nature of the boundary surface of the solid after strain, otherwise we cannot satisfy the boundary conditions. We, therefore, assume that the equation of the boundary of the strained solid is known. In small displacements it does not affect the results whether we use the boundary of the strained or unstrained solid, but not so for finite displacement, *e.g.*, a straight rubber tubing bent into the form of an anchor ring.

4—Voigt's *Extension of the Strain-Energy Function*

If Hooke's Law is regarded as a first approximation valid for very small strains, it is natural to assume that the terms of the second order in the strain-energy function also constitute a first approximation. For a second approximation we should include terms of the third order in the invariants of the strain quadric. An attempt has been made in this direction by Voigt.* He takes the three invariants,

$$\delta = S_x + S_y + S_z,$$
$$\xi = S_y S_z + S_z S_x + S_x S_y - \tfrac{1}{4}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2),$$
$$\zeta = S_x S_y S_z + \tfrac{1}{4}\sigma_{xy}\sigma_{yz}\sigma_{zx} - \tfrac{1}{4}(S_x \sigma_{yz}^2 + S_y \sigma_{zx}^2 + S_z \sigma_{xy}^2),$$

of the strain quadric, and putting

$$I = \delta^2 - 2\xi,$$

constructs the new energy function

$$2W = c_1 \delta^2 + c_2 I + \tfrac{2}{3} c'_1 \delta^3 + c'_2 I\delta + 2c'_3 \zeta,$$

c'_1, c'_2, c'_3 being three new elastic constants. In applying his method to a few particular cases, he assumes that the displacements given by the present theory are a first approximation. For example, in treating the problem of the torsion of a circular cylinder, he takes

$$u = -\tau yz + u',$$
$$v = \tau zx + v',$$
$$w = w',$$

u', v', w' being the corrections to be added for a second approximation. Assuming u', v', w' to be of the second degree in τ, he calculates the stresses from the modified energy function neglecting all powers of τ higher than the second. It is obvious that this method cannot be justified for finite displacements.

5—*Justification of our Assumption*

In general, the elastic behaviour of a material has reference to certain directions fixed relatively to the material. If, however, the material is isotropic the formulæ

* 'Ann. Physik,' vol. 52, p. 536 (1894).

connecting stress-components with strain-components are independent of direction. Since we assume throughout this paper that the elastic body to be considered is always isotropic any assumption that we may make with regard to the stress-strain relations must be such that these relations are an invariant for all transformations from one set of orthogonal axes to another. In other words, these relations must be in tensor form. Now we know that this condition is satisfied if we take

$$\widehat{xx} = \lambda\delta + 2\mu S_x,$$
$$\widehat{yz} = \mu\sigma_{yz}, \text{ etc.},$$

where S_x, σ_{yz} are now given by (1). Since this is the simplest tensor form that we can take, it is quite natural for us to assume that the stress-strain relations are governed by equations of the above type.

6—*Object of the Paper*

The object of this paper is to discuss the following three particular cases in the light of what has been said in the foregoing pages.

(1) A cylinder subjected to a uniform tension T.
(2) A rectangular plate bent into the form of a right cylindrical surface.
(3) Torsion of a right circular cylinder.

Virtually, SAINT-VENANT's *semi-inverse* method has been used in all the three cases. Frequent applications have also been made of his "principle of the elastic equivalence of statically equipollent systems of load."

II—CYLINDER SUBJECTED TO A UNIFORM TENSION T

7—*Comparison with experimental results*

Various authors have drawn attention to the fact that the elongation produced in a wire is not always proportional to the stretching force even when we remain within the limits of perfect elasticity. GERSTNER* assumed an empirical relation of the form

$$T = As + Bs^2 + Cs^3 + Ds^4 + \dots, \qquad (3)$$

T being the tension and s the total stretch. His experiments led him to the conclusion that C, D, etc., are all zero, and that B is negative. Later in 1844, HODGKINSON† confirmed this result. The names of WERTHEIM‡ and THOMPSON§

* "Handbuch der Mechanik," Part I, p. 241 (1831).
† "Rep. Brit. Ass.," Part 2, pp. 25-27 (1844).
‡ 'Ann. Chim. (Phys.),' vol. 21, p. 385 (1847).
§ 'Ann. Physik,' vol. 44, p. 555 (1891).

must also be mentioned in this connection. From his experiments THOMPSON concludes that a relation of the form,

$$s = aT + bT^2 + cT^3, \quad \ldots \ldots \ldots \ldots \quad (4)$$

fits in best with his results. If we solve GERSTNER's relation $T = As + Bs^2$ and neglect powers of (B/A^2) higher than the third we get an equation of the form given in (4). Without going any further into the details of the experiments conducted by these authors let us see what result our form of analysis is going to give.

Consider

$$u = x - px,$$
$$v = y - qy,$$
$$w = z - rz,$$

p, q, r being constants. The generalized strain components are given by

$$S_x = \tfrac{1}{2}(1 - p^2),$$
$$S_y = \tfrac{1}{2}(1 - q^2),$$
$$S_z = \tfrac{1}{2}(1 - r^2),$$
$$\sigma_{yz} = \sigma_{zx} = \sigma_{xy} = 0.$$

Therefore

$$\delta = S_x + S_y + S_z = \tfrac{1}{2}[3 - (p^2 + q^2 + r^2)].$$

Since all the components of stress are constant, they satisfy the body-stress equations. Again we have assumed that the cylinder has any form of boundary whatever. We must, therefore, have $\widehat{xx} = \widehat{yy} = 0$. Thus we get

$$\lambda\delta + 2\mu S_x = \lambda\delta + 2\mu S_y = 0,$$

or

$$p^2 = q^2, \quad \ldots \ldots \ldots \ldots \quad (5.1)$$

and

$$(3\lambda + 2\mu) - 2(\lambda + \mu)p^2 - \lambda r^2 = 0. \quad \ldots \ldots \quad (5.2)$$

Again, if T is the uniform tension applied to the plane ends of the cylinder, we have

$$2T = (3\lambda + 2\mu) - 2\lambda p^2 - (\lambda + 2\mu)r^2,$$

and using (5.2) we get

$$T = \tfrac{1}{2}E(1 - r^2) = ES_z,$$

which was to be expected from the stress-strain relations.

Therefore

$$r = \pm\left(1 - \frac{2T}{E}\right)^{\frac{1}{2}}. \quad \ldots \ldots \ldots \ldots \quad (6)$$

To decide the sign we observe that for a simple tension z is always positive or negative according as z', the co-ordinate before strain is positive or negative. Since $z' = rz$, we must take the positive sign with the square root in (6). This is also confirmed by the fact that if we neglect powers of T/E higher than the first we get $w = \mathrm{TZ/E}$, a result true for small strains.

From (5.2) we have

$$p = \pm\left(1+\frac{2\eta \mathrm{T}}{\mathrm{E}}\right)^{\frac{1}{2}},$$

and neglecting the negative sign as before, we get

$$p = q = \left(1+\frac{2\eta \mathrm{T}}{\mathrm{E}}\right)^{\frac{1}{2}}.$$

Thus the only physically possible solution is given by

$$u = x\left[1-\left(1+\frac{2\eta \mathrm{T}}{\mathrm{E}}\right)^{\frac{1}{2}}\right]$$

$$v = y\left[1-\left(1+\frac{2\eta \mathrm{T}}{\mathrm{E}}\right)^{\frac{1}{2}}\right]$$

$$w = z\left[1-\left(1-\frac{2\mathrm{T}}{\mathrm{E}}\right)^{\frac{1}{2}}\right].$$

It appears, therefore, that the longitudinal displacement measured per unit length of the length of an extended fibre is not proportional to T/E as in the approximate theory. If we assume that s is the ordinary longitudinal stretch we easily get from (6) and the relation $\mathrm{T} = \mathrm{ES}_z$

$$2\mathrm{T} = \mathrm{E}\left\{1-\frac{1}{(1+s)^2}\right\}$$

$$= \mathrm{E}\,(2s - 3s^2 + 4s^3 \ldots). \qquad \text{(A)}$$

If s^3, s^4, s^5, etc., be too small to be taken into account, we see that the above relation reduces to the same form as assumed and experimentally verified both by Gerstner and Hodgkinson. It is interesting to observe that the conclusion that the coefficient B of s^2 in (3) is negative is confirmed by this result.

From the above equation connecting T and s, we also see that $\mathrm{T} = \frac{1}{2}\mathrm{E}$ leads to an infinite elongation and therefore roughly corresponds to the yield point. Apparently no material can stand a tension of this amount. It follows that the W-displacement, and hence the solution, is always real if we remain within the limits of perfect elasticity.

The stress-strain curve given by (A) is very suggestive and not unlike that which is actually found in some materials. In fig. 1 we have plotted this curve by taking S as the abscissa and T/E as the ordinate.

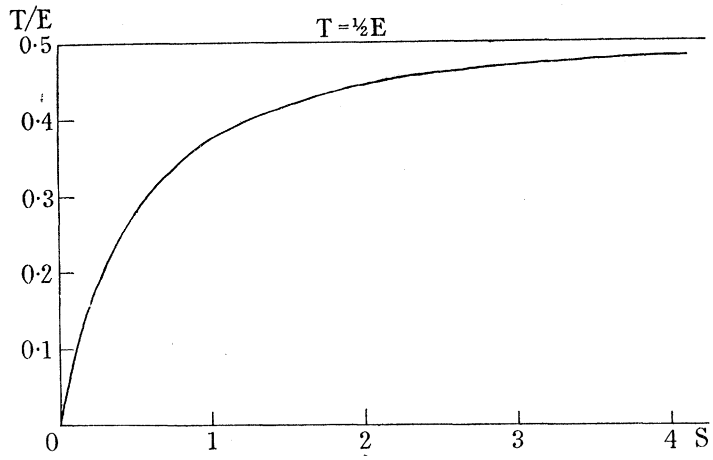

FIG. 1

III—BENDING OF A RECTANGULAR PLATE IN THE FORM OF A RIGHT CYLINDRICAL SURFACE

8—*History of the Problem*

It is a well-known fact that long beams can be bent by terminal couples without producing any anti-clastic curvature in them. In all such cases the displacements are not in any way small, and the curvature produced by the applied couples is considerable. Hence it is that the stress-distribution varies from that indicated by BERNOULLI-EULER theory. KELVIN and TAIT* have drawn attention to the problem of the " flexure of a broad very thin band (such as a watch spring) bent into a circle of radius comparable with a third proportional to its thickness and breadth." LAMB† has given an approximate solution of this problem based on the theory of small strains. The drawback in the solution is that it cannot be applied to those physically possible cases where u, v, w are finite.

LOVE‡ has shown that if the thickness of the plate is small enough, or the radius of curvature great enough, the plate can be bent into a cylindrical form by two couples applied at its edges only. Without imposing any such restrictions we propose to solve the same problem in the following pages. We shall also show that the two applied couples reduce to the values obtained by LOVE if the thickness of the plate be small.

* 'Natural Philosophy,' Camb., vol. 2, pp. 264-265 (1886).
† 'Mem. Manchr. Lit. Phil. Soc.,' vol. 3, p. 216 (1890) ; also 'Phil. Mag.,' vol. 31, p. 182 (1891).
‡ " Mathematical Theory of Elasticity," p. 554, 4th ed.

9—*Tentative Values for the Displacement*

We suppose that an initially plane rectangular plate is bent into the form of a circular cylinder with two edges as generators, and seek the forces that must be applied to it to hold it in this form.

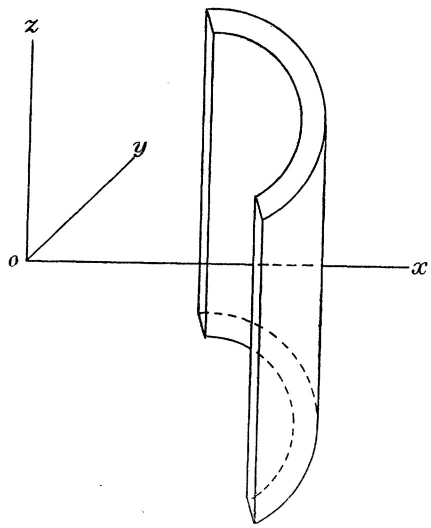

FIG. 2

Let (x', y', z') be the co-ordinates of a point P of the plate before strain. We assume that two faces of the plate get bent into right cylindrical surfaces of inner radius a and outer radius b, and that the other two into axial terminating planes given by $\theta = \pm z$. The axis of z' is parallel to the axis of the cylinder, which we take as the axis of z, see fig. 2.

From the symmetrical manner in which the plate is strained, it appears—

(1) that a cross-section perpendicular to the axis of z' remains plane after strain;
(2) that every fibre parallel to the axis of z' is extended, if at all, by the same amount;
(3) that the planes given by x' = constant get bent into right cylindrical surfaces with the z — axis as their common axis.

These tentative considerations lead us to assume

$$u = x - f(r), \qquad (7.1)$$

$$v = y - \phi(x, y), \qquad (7.2)$$

$$w = \alpha z, \qquad (7.3)$$

where $f(r)$ is a function of $r = (x^2 + y^2)^{\frac{1}{2}}$ only, $\phi(x, y)$ an undetermined function of (x, y), and α a constant.

10—*Components of Strain and Stress*

From (1) we have

$$S_x = \tfrac{1}{2} - \tfrac{1}{2}\left[\frac{f'^2 x^2}{r^2} + \left(\frac{\partial \phi}{\partial x}\right)^2\right], \qquad (8.1)$$

$$S_y = \tfrac{1}{2} - \tfrac{1}{2}\left[\frac{f'^2 y^2}{r^2} + \left(\frac{\partial \phi}{\partial y}\right)^2\right], \qquad (8.2)$$

$$S_z = \alpha - \tfrac{1}{2}\alpha^2, \qquad (8.3)$$

$$\sigma_{xz} = \sigma_{yz} = 0, \qquad (8.4)$$

$$\sigma_{xy} = -\left[\frac{f'^2 xy}{r^2} + \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\right], \qquad (8.5)$$

where

$$f' = \frac{df}{dr}.$$

Therefore

$$\begin{aligned}\delta &= S_x + S_y + S_z, \\ &= K - \tfrac{1}{2}\left[f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right], \quad \ldots \quad (9)\end{aligned}$$

where

$$K = 1 + \alpha - \tfrac{1}{2}\alpha^2.$$

The stresses are now given by

$$\widehat{xx} = \lambda\left[K - \tfrac{1}{2}\left\{f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right] + \mu\left[1 - \frac{f'^2x^2}{r^2} - \left(\frac{\partial\phi}{\partial x}\right)^2\right], \quad (10.1)$$

$$\widehat{yy} = \lambda\left[K - \tfrac{1}{2}\left\{f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right] + \mu\left[1 - \frac{f'^2y^2}{r^2} - \left(\frac{\partial\phi}{\partial y}\right)^2\right], \quad (10.2)$$

$$\widehat{zz} = \lambda\left[K - \tfrac{1}{2}\left\{f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right] + 2\mu\,(K - 1), \quad \ldots \quad (10.3)$$

$$\widehat{yz} = \widehat{zx} = 0, \quad \ldots \quad (10.4)$$

$$\widehat{xy} = -\mu\left[\frac{f'^2xy}{r^2} + \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right]. \quad \ldots \quad (10.5)$$

11—*Satisfaction of the Body-Stress Equations and Boundary Conditions*

These stresses have to satisfy the body-stress equations. The equation,

$$\frac{\partial\widehat{xx}}{\partial x} + \frac{\partial\widehat{xy}}{\partial y} = 0,$$

gives

$$\tfrac{1}{2}\lambda\frac{\partial}{\partial x}\left[f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \mu\left[\frac{\partial}{\partial x}\left\{\frac{f'^2x^2}{r^2} + \left(\frac{\partial\phi}{\partial x}\right)^2\right\} + \frac{\partial}{\partial y}\left\{\frac{f'^2xy}{r^2} + \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right\}\right] = 0,$$

or

$$\frac{\partial}{\partial x}\left[\tfrac{1}{2}\lambda\left\{f'^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} + \mu\left\{f'^2 + \int\frac{f'^2}{r}\,dr + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right]$$
$$+ \mu\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right]\frac{\partial\phi}{\partial x} = 0. \quad (11.1)$$

Similarly

$$\frac{\partial\widehat{xy}}{\partial x} + \frac{\partial\widehat{yy}}{\partial y} = 0$$

gives

$$\frac{\partial}{\partial y}\left[\tfrac{1}{2}\lambda\left\{f'^2+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}+\mu\left\{f'^2+\int\frac{f'^2}{r}\,dr+\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right]$$
$$+\mu\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]\frac{\partial\phi}{\partial y}=0, \qquad (11.2)$$

and the third body-stress equation is identically satisfied. Differentiating (11.1) with respect to y and (11.2) with respect to x and then subtracting one from the other, we get the equation satisfied by ϕ as

$$\frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial x}\nabla^2\phi\right]=\frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial y}\nabla^2\phi\right], \quad \dots\dots\dots \quad (12)$$

or

$$\nabla^2\phi = \text{a function of } \phi.$$

Let us now turn to the boundary conditions. The first

$$x\,.\,\widehat{xx}+y\,.\,\widehat{xy}=0$$

is to hold good both over $r=a$ and $r=b$. Substituting the values of $\widehat{xx}$, $\widehat{xy}$ from (10), we have

$$x\left[\lambda K-\tfrac{1}{2}\lambda\left\{f'^2+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}+\mu\,(1-f'^2)\right]-\mu\frac{\partial\phi}{\partial x}\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)=0, \quad \dots\dots \quad (13.1)$$

and the second in like manner gives

$$y\left[\lambda K-\tfrac{1}{2}\lambda\left\{f'^2+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}+\mu\,(1-f'^2)\right]-\mu\frac{\partial\phi}{\partial y}\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)=0. \quad \dots\dots \quad (13.2)$$

Hence we must have on the boundary

$$\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)\left(y\frac{\partial\phi}{\partial x}-x\frac{\partial\phi}{\partial y}\right)=0, \quad \dots\dots\dots \quad (14)$$

i.e., either $\dfrac{\partial\phi}{\partial r}=0$ or $\dfrac{\partial\phi}{\partial\theta}=0$.

But $y'=\phi\,(x,y)$ from (7.2), and since the planes $y'=$ constant cannot also get bent into $r=$ constant we must take

$$\phi = A\theta + B, \quad \dots\dots\dots\dots \quad (15)$$

where A and B are constants. This value of ϕ also satisfies (12).

Putting the value of ϕ given by (15) in (11.1) and (11.2), we see that both of them reduce to the same differential equation, viz.,

$$(\tfrac{1}{2}\lambda+\mu)\,f'^2+\mu\int\frac{f'^2}{r}\,dr+\tfrac{1}{2}\,(\lambda+\mu)\,\frac{A^2}{r^2}=\text{a constant}. \quad \dots \quad (16)$$

Putting $f'^2 = \beta$ and differentiating (16) with respect to r, we have

$$(\tfrac{1}{2}\lambda+\mu)\frac{d\beta}{dr}+\frac{\mu\beta}{r}=(\lambda+\mu)\frac{A^2}{r^3},$$

which is a linear equation in β, whose solution is

$$\beta = f'^2 = \frac{l}{r^{2\mu/(\lambda+2\mu)}} - \frac{A^2}{r^2}, \qquad (17)$$

l being a constant of integration.

The boundary conditions (13.1) and (13.2) both reduce to

$$\lambda\left[K-\tfrac{1}{2}\left(f'^2+\frac{A^2}{r^2}\right)\right]+\mu\,(1-f'^2)=0. \qquad (18)$$

Substituting the value of f'^2 from (17) we get the two equations

$$(\lambda K+\mu)+\frac{\mu A^2}{a^2}-\frac{(\tfrac{1}{2}\lambda+\mu)\,l}{a^{2\mu/(\lambda+2\mu)}}=0, \qquad (19.1)$$

and

$$(\lambda K+\mu)+\frac{\mu A^2}{b^2}-\frac{(\tfrac{1}{2}\lambda+\mu)\,l}{b^{2\mu/(\lambda+2\mu)}}=0. \qquad (19.2)$$

Let us put $c=(1-2\eta)/(1-\eta)$, where η is the Poisson's ratio. Since $\eta > \frac{1}{2}$ it is clear that c always lies between 0 and 1. In the extreme case of $\eta=\frac{1}{2}$ we have $c=0$. From (19) we now have

$$\mu A^2=\frac{(\lambda K+\mu)\,(ab)^{2-c}\,(a^c-b^c)}{a^{2-c}-b^{2-c}}, \qquad (20.1)$$

$$(\tfrac{1}{2}\lambda+\mu)\,l=\frac{(\lambda K+\mu)\,(a^2-b^2)}{a^{2-c}-b^{2-c}}. \qquad (20.2)$$

As for the constant B in (15) we can assume that $y'=0$ for $\theta=0$ so that $B=0$. The constant K has still to be determined.

12—*The Bending Couples* M_1 *and* M_2

Let us now consider the tractions across a plane θ = constant. For this we first calculate the values of $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{\theta\theta}$. They can easily be seen to be given by

$$\widehat{rr}=\lambda K+\mu+\frac{\mu A^2}{r^2}-\frac{(\tfrac{1}{2}\lambda+\mu)\,l}{r^c}, \qquad (21.1)$$

$$\widehat{r\theta}=0, \qquad (21.2)$$

$$\widehat{\theta\theta}=\lambda K+\mu-\frac{\mu A^2}{r^2}-\tfrac{1}{2}\frac{\lambda l}{r^c}. \qquad (21.3)$$

If n denotes the direction of a normal to $\theta =$ constant we have the tractions across it given by

$$\widehat{rn} = 0,$$
$$\widehat{\theta n} = \widehat{\theta\theta}.$$

Again, if the beam is bent by terminal couples only we must have

$$\int_a^b \widehat{\theta\theta}\, dr = 0,$$

i.e.,

$$(\lambda K + \mu)(b - a) + \mu A^2 \left(\frac{1}{b} - \frac{1}{a}\right) - l\,(\tfrac{1}{2}\lambda + \mu)(b^{1-c} - a^{1-c}) = 0.$$

Multuplying (19.1) by a and (19.2) by b and subtracting one from the other, we see that this condition is satisfied. Accordingly the resultant traction across $\theta =$ constant reduces to a couple.

If M_1 be the moment per unit length of this couple about the z — axis,

$$M_1 = \int_a^b r\, \widehat{\theta\theta}\, dr$$
$$= \tfrac{1}{2}(\lambda K + \mu)(b^2 - a^2) - \mu A^2 (\log b - \log a) - \frac{\lambda l}{2\,(2 - c)}(b^{2-c} - a^{2-c}). \quad (22)$$

We have still to consider the tractions across a plane-section given by $z =$ constant. Since $\widehat{yz}$ and $\widehat{xz}$ are both zero,

$$\widehat{xn} = \widehat{yn} = 0,$$
$$\widehat{zn} = \widehat{zz}.$$

If $-M_2$ be the elementary couple in the axial plane applied to the plane ends of the cylinder between θ and $\theta + d\theta$, we must have

$$\int_a^b r\, \widehat{zz}\, dr = 0, \qquad \int_a^b r^2\, \widehat{zz}\, dr = -M_2. \quad (23)$$

The first of these gives

$$\tfrac{1}{2}[(\lambda + 2\mu) K - 2\mu](b^2 - a^2) - \frac{\lambda(\lambda + 2\mu)\, l}{4\,(\lambda + \mu)}(b^{2-c} - a^{2-c}) = 0,$$

and putting the value of l from (20.2) we get

$$(\lambda + 2\mu) K - 2\mu - \frac{\lambda(\lambda K + \mu)}{\lambda + \mu} = 0,$$

which gives $K = 1$.

Hence

$$1 + \alpha - \tfrac{1}{2}\alpha^2 = 1,$$

i.e.,

$$\alpha = 0, \quad \text{or} \quad \alpha = 2.$$

$\alpha = 2$ gives $w = 2z$, and, therefore, $z = -z'$. This is a physically impossible solution, for the present problem* and hence must be neglected. $\alpha = 0$ only means that fibres parallel to the z' — axis suffer no change.

The second of equations (23) gives

$$M_2 = -\lambda \int_a^b r^2 \left(1 - \tfrac{1}{2} \frac{l}{r^c}\right) dr$$

$$= -\lambda \left[\tfrac{1}{3}(b^3 - a^3) + \tfrac{1}{2} \frac{2-c}{3-c}(a^2 - b^2) \frac{(a^{3-c} - b^{3-c})}{a^{2-c} - b^{2-c}}\right] \quad . \quad . \quad . \quad (24)$$

13—*Determination of f and the Neutral Axis*

To determine f we go back to equation (17). We have

$$\frac{df}{dr} = \pm \left(\frac{l}{r^c} - \frac{A^2}{r^2}\right)^{\frac{1}{2}}.$$

If we assume that x' increases with r we see that we should take the positive sign with the square root on the right-hand side of the above equation.† Putting

$$l^{\frac{1}{2}} r^{\frac{1}{2}(2-c)} = A \operatorname{Sec} \psi,$$

we get

$$(2-c) f = 2A(\tan \psi - \psi) + \text{a constant}$$

$$= 2\left[(lr^{2-c} - A^2)^{\frac{1}{2}} - A \operatorname{Sec}^{-1}\left(\frac{lr^{2-c}}{A^2}\right)^{\frac{1}{2}}\right] + \text{a constant}.$$

The general value of ψ is given by

$$\psi = 2n\pi \pm \psi_0 \qquad \text{or} \qquad 2n\pi \pm (\pi - \psi_0),$$

according as $\sqrt{l}$ is taken with a positive or a negative sign, ψ_0 being the principal positive value of ψ. Now we can always assume that $x' = f$ is positive. We have already taken $y' = A\theta$. The constant in the value of f only affects the original distance of the plate from the origin, and hence can be assumed to be zero. Thus we can write

$$f = \frac{2A}{2-c}(\tan \psi_0 - \psi_0)$$

$$= 2\left[\frac{(a^c - b^c)(ab)^{2-c}}{c(2-c)(a^{2-c} - b^{2-c})}\right]^{\frac{1}{2}} \left[\left\{\frac{cr^{2-c}(a^2 - b^2)}{(ab)^{2-c}(a^c - b^c)} - 1\right\}^{\frac{1}{2}} - \operatorname{Sec}^{-1}\left\{\frac{cr^{2-c}(a^2 - b^2)}{(ab)^{2-c}(a^c - b^c)}\right\}^{\frac{1}{2}}\right].$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

* If the cylinder is turned upside down we get $w = 2z$. Accompanied by a change of sign of A given by (20.1) it only amounts to a rigid body displacement.

† Taking the negative sign with the value of df/dr only amounts to bending the plate in the opposite direction.

From (8) we easily get

$$S_\vartheta = \tfrac{1}{2}\left(1 - \frac{A^2}{r^2}\right), \qquad (26)$$

which shows that $r = A$ is the unstretched longitudinal fibre.

From (21) we have the principal stress difference

$$\widehat{\theta\theta} - \widehat{rr} = \mu\left(\frac{l}{r^c} - \frac{2A^2}{r^2}\right).$$

This does not vanish over the neutral axis $(r = A)$ since $l \neq 2A^c$.

14—M_1 *and* M_2 *for a Thin Plate*

From (22) we have

$$M_1 = \mu\left[\tfrac{1}{2}(b^2 - a^2) - \frac{(2-c)(\log b - \log a)(ab)^{2-c}(a^c - b^c)}{c(a^{2-c} - b^{2-c})}\right]. \qquad (27)$$

If the rectangular plate is very thin we can put $b = a + h$, where h is very small. In such a case M_1 is given by

$$M_1 = \mu\left[\tfrac{1}{2}(h^2 + 2ah) - \frac{2-c}{c}\cdot a^2\left(1 + \frac{h}{a}\right)^{2-c}\frac{1 - (1 + h/a)^c}{1 - (1 + h/a)^{2-c}}\cdot \log\left(1 + \frac{h}{a}\right)\right].$$

Let us neglect all powers of h higher than the third, then

$$M_1 = \mu\left[\tfrac{1}{2}(h^2 + 2ah) - \frac{2-c}{c}a^2\left\{1 + (2-c)\frac{h}{a} + \tfrac{1}{2}(2-c)(1-c)\frac{h^2}{a^2}\right\}\right.$$
$$\left.\times\left(\frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{3a^3}\right)\frac{1 - (1 + h/a)^c}{1 - (1 + h/a)^{2-c}}\right]. \qquad (28)$$

Now

$$\frac{1 - (1 + h/a)^c}{1 - (1 + h/a)^{2-c}} = \frac{c}{2-c}\cdot\frac{1 + \frac{1}{2}(c-1)(h/a) + \frac{1}{6}(c-1)(c-2)(h^2/a^2)}{1 + \frac{1}{2}(1-c)(h/a)(1 - \frac{1}{3}ch/a)}.$$

We know that $c < 1$. So is h/a, therefore

$$\tfrac{1}{2}(1-c)(h/a) < 1,$$

and also

$$(1 - \tfrac{1}{3}ch/a) < 1.$$

Therefore

$$\tfrac{1}{2}(1-c)(h/a)(1 - \tfrac{1}{3}ch/a) < 1.$$

Hence we can expand

$$[1 + \tfrac{1}{2}(1-c)(h/a)(1 - \tfrac{1}{3}ch/a)]^{-1}$$

in powers of h/a.

The above identity now becomes

$$\frac{2-c}{c}\cdot\frac{1-(1+h/a)^{c}}{1-(1+h/a)^{2-c}}=\left[1+\tfrac{1}{2}(c-1)\frac{h}{a}+\tfrac{1}{6}(c-1)(c-2)\frac{h^2}{a^2}\right]$$
$$\times\left[1-\tfrac{1}{2}(1-c)\frac{h}{a}+\tfrac{1}{6}c(1-c)\frac{h^2}{a^2}+\tfrac{1}{4}(1-c)^2\frac{h^2}{a^2}\right]$$
$$=1-(1-c)\frac{h}{a}+\tfrac{1}{2}(1-c)(\tfrac{5}{3}-c)\frac{h^2}{a^2}.$$

We can now rewrite (28) as

$$M_1=\mu\left[\tfrac{1}{2}(h^2+2ah)-ah\left(1-\tfrac{1}{2}\frac{h}{a}+\tfrac{1}{3}\frac{h^2}{a^2}\right)\left\{1+\frac{h}{a}-\tfrac{1}{6}(1-c)\frac{h^2}{a^2}\right\}\right]$$
$$=\tfrac{1}{6}\mu(2-c)\frac{h^3}{a}.$$

The factor $\frac{1}{6}\mu(2-c)h^3$ is what Love calls D in his treatment of the problem of plates. Thus we get

$$M_1=\frac{D}{a}, \qquad (29)$$

a result which is the same as obtained by Love* for small strains.

Again, from (24) we have

$$M_2=-\lambda\left[\tfrac{1}{3}h(3a^2+3ah+h^2)+\tfrac{1}{2}\cdot\frac{2-c}{3-c}\cdot a(h^2+2ah)\cdot\frac{1-(1+h/a)^{3-c}}{1-(1+h/a)^{2-c}}\right],$$

and proceeding as above we get

$$M_2=-\tfrac{1}{12}\lambda ch^3=-\frac{\eta D}{a}$$

in Love's notation.

15—*Values of u, v, w, $\widehat{xx}$, $\widehat{yz}$, etc.*

To recapitulate we have the displacement (u, v, w) given by

$$u=x-\frac{2A}{2-c}(\tan\psi_0-\psi_0),$$
$$v=y-A\theta,$$
$$w=0,$$

where A is the positive value given by (20.1), and ψ_0 is the principal positive value of

$$l^{\frac{1}{2}}r^{\frac{1}{2}(2-c)}=A\sec\psi,$$

* *Op. cit.*, p. 554.

$l^{\frac{1}{2}}$ being the positive value given by (20.2), and the components of stress by

$$\widehat{xx} = \lambda + \mu - \tfrac{1}{2}\lambda \frac{l}{r^c} - \frac{\mu}{r^2}\left[\frac{lx^2}{r^c} - \frac{A^2}{r^2}(x^2 - y^2)\right],$$

$$\widehat{yy} = (\lambda + \mu) - \tfrac{1}{2}\lambda \frac{l}{r^c} - \frac{\mu}{r^2}\left[\frac{ly^2}{r^c} + \frac{A^2}{r^2}(x^2 - y^2)\right],$$

$$\widehat{zz} = \lambda\left(1 - \tfrac{1}{2}\frac{l}{r^c}\right),$$

$$\widehat{yz} = \widehat{zx} = 0,$$

$$\widehat{xy} = -\frac{\mu xy}{r^2}\left[\frac{l}{r^c} - \frac{2A^2}{r^2}\right].$$

The principal stresses are given by

$$\widehat{rr} = \lambda + \mu + \frac{\mu A^2}{r^2} - (\tfrac{1}{2}\lambda + \mu)\frac{l}{r^c},$$

$$\widehat{r\theta} = 0,$$

$$\widehat{\theta\theta} = \lambda + \mu - \frac{\mu A^2}{r^2} - \tfrac{1}{2}\lambda\frac{l}{r^c}.$$

The v — displacement shows that all planes $y' =$ constant have changed into planes passing through the axis of the cylinder. This is quite a natural thing to expect. In fact, from a physical point of view it is the simplest change which can happen to a plate bent into a cylinder.

IV—Torsion of a Right Circular Cylinder

16—*Need for Extension of the Present Theory*

The necessity of calculating the torsional couple on a circular cylinder to a greater degree of accuracy than what exists in its present well-known value, $\frac{1}{2}\mu\tau\pi a^4$, has been recognized by various authors. Saint-Venant himself has calculated the correction which should be added to it if powers of τ higher than the second can be neglected. To this we shall refer later. One of the most important cases in which τ^2, and possibly a few more of its higher powers cannot be neglected, is that of the conduction of heat and electricity along twisted wires. Recent experiments by Lees and Calthrop* on wires made of steel, copper, aluminium, etc., have shown that the effect of twist on them is to decrease both their heat conductivity and electrical conductivity. This decrease is independent of the direction of the twist and is proportional to τ^2 for small values of τ. The earlier experiments of Johnstone† show that when a wire is *stretched* the heat conductivity along it is slightly increased. Thus arises the need of determining axial stresses, neglected in the approximate theory, in cylinders under torsion.

* 'Proc. Phys. Soc.,' vol. 35, p. 225 (1923).

† 'Proc. Phys. Soc.,' vol. 29, p. 195 (1917).

Balinkin* has carried out an optical determination of these stresses for long rectangular plates. Later in the paper we shall have occasion to point out that the theoretical discussion given by him is not very accurate.

17—Saint-Venant *on Large Torsional Shifts*

Saint-Venant himself has given a modified form of his well-known solution of the torsional problem for large torsional shifts. On p. 347 of his famous memoir† he says that the displacements

$$u = -\tau yz,$$
$$v = \tau xz,$$
$$w = \tau\phi,$$

do not hold good for large torsional shifts. By an easy process of summation he finds the new values as

$$u = -y \sin \tau z + x (1 - \cos \tau z),$$
$$v = x \sin \tau z + y (1 - \cos \tau z),$$
$$w = \tau\phi,$$

ϕ, the torsion function remaining unchanged, and x, y, and z being the co-ordinate of a point P in the strained solid. The above values of the displacement give a correct solution only if the second order terms in the components of strain are neglected. Presently we shall show that for a right-circular cylinder (u, v, w) should be taken as

$$u = x (1 - \beta \cos \tau z) - y\beta \sin \tau z,$$
$$v = y (1 - \beta \cos \tau z) + x\beta \sin \tau z,$$
$$w = \alpha z,$$

where β is a function of $r = (x^2 + y^2)^{\frac{1}{2}}$ only, and α is a constant. These reduce to the values given by Saint-Venant if we take $\beta = 1$ and $\alpha = 0$. But on calculation, after using the values of the strain components given by (1), we find that the body-stress equations are not satisfied if we take Saint-Venant's values. In fact, we get

$$\frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{xy}}{\partial y} + \frac{\partial \widehat{xz}}{\partial z} = -\lambda\tau^2 x,$$
$$\frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} + \frac{\partial \widehat{yz}}{\partial z} = -\lambda\tau^2 y,$$
$$\frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} + \frac{\partial \widehat{xz}}{\partial z} = 0.$$

Since we have assumed the strain to be finite we cannot use Saint-Venant's values of the component displacements.

* 'Phys. Rev.,' vol. 30, p. 520 (1927).
† 'Mém. des Savants Étrangers,' vol. 14 (1855).

18—*Finite Components of Displacement*

We shall now calculate the displacements when a right circular cylinder is subjected to a finite twist τ. From considerations of symmetry we know

(I) that cross-sections must remain plane;

(II) that straight radii must remain straight.

Moreover, the displacement w, if the conditions are uniform, must be of the form αz. The displacement in the cross-section, therefore, must consist of a rotation τz, about the axis of z and a radial elongation which must be a function of $r = (x^2 + y^2)^{\frac{1}{2}}$ only.

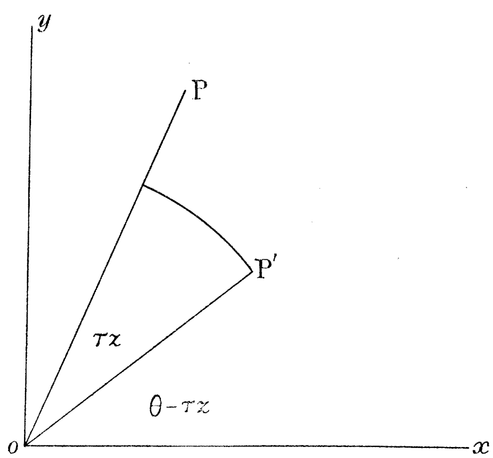

FIG. 3

Thus accented co-ordinates referring to the initial positions

$$x' = \mathrm{OP}' \cos(\theta - \tau z) = \beta r (\cos\theta \cos\tau z + \sin\theta \sin\tau z) = \beta (x \cos\tau z + y \sin\tau z),$$

$$y' = \mathrm{OP}' \sin(\theta - \tau z) = \beta r (\sin\theta \cos\tau z - \cos\theta \sin\tau z) = \beta (y \cos\tau z - x \sin\tau z),$$

where β is a function of r only.

Therefore

$$u = x - x' = x(1 - \beta\cos\tau z) - y\beta\sin\tau z, \quad (30.1)$$

$$v = y - y' = y(1 - \beta\cos\tau z) + x\beta\sin\tau z, \quad (30.2)$$

$$w = \alpha z. \quad (30.3)$$

19—*Calculation of Strains and Stresses*

We now calculate the strains. Substituting the values of u, v, w from (30) in (1) we get

$$S_x = \tfrac{1}{2}(1 - \beta^2) - \tfrac{1}{2}x^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \quad (31.1)$$

$$S_y = \tfrac{1}{2}(1 - \beta^2) - \tfrac{1}{2}y^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \quad (31.2)$$

$$S_z = \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2, \quad (31.3)$$

$$\sigma_{yz} = \tau\beta^2 x, \quad (31.4)$$

$$\sigma_{zx} = -\tau\beta^2 y, \quad (31.5)$$

$$\sigma_{xy} = -xy\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \quad (31.6)$$

where $\beta' = d\beta/dr$.

The stresses are given by

$$\widehat{xx} = \lambda\left[1-\beta^2-\tfrac{1}{2}r^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)+\alpha-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}\beta^2\tau^2r^2\right] + \mu\left[1-\beta^2-x^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right], \quad (32.1)$$

$$\widehat{yy} = \lambda\left[1-\beta^2-\tfrac{1}{2}r^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)+\alpha-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}\beta^2\tau^2r^2\right] + \mu\left[1-\beta^2-y^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right], \quad (32.2)$$

$$\widehat{zz} = \lambda\left[1-\beta^2-\tfrac{1}{2}r^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)+\alpha-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}\beta^2\tau^2r^2\right] + \mu\,(2\alpha-\alpha^2-\beta^2\tau^2r^2), \quad (32.3)$$

$$\widehat{yz} = \mu\tau\beta^2x, \quad (32.4)$$

$$\widehat{zx} = -\mu\tau\beta^2y, \quad (32.5)$$

$$\widehat{xy} = -\mu xy\left(\beta'^2+\frac{2\beta\beta'}{r}\right). \quad (32.6)$$

20—*Satisfaction of the Body-Stress Equations*

These stresses have to satisfy the body-stress equations. The first body-stress equation,

$$\frac{\partial\widehat{xx}}{\partial x}+\frac{\partial\widehat{xy}}{\partial y}+\frac{\partial\widehat{xz}}{\partial z}=0,$$

gives

$$\frac{\partial}{\partial x}\left[(\lambda+\mu)\,\beta^2+\tfrac{1}{2}\beta^2\tau^2r^2\lambda+(\tfrac{1}{2}\lambda r^2+\mu x^2)\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right] + \frac{\partial}{\partial y}\left[\mu xy\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right]=0,$$

which can be rewritten as

$$\frac{\partial}{\partial x}\left[(\lambda+\mu)\,\beta^2+\tfrac{1}{2}\beta^2\tau^2r^2\lambda+(\tfrac{1}{2}\lambda+\mu)\left(\beta'^2+\frac{2\beta\beta'}{r}\right)r^2+\mu\int(r\beta'^2+2\beta\beta')\,dr\right]=0,$$

or as

$$\frac{\partial}{\partial x}\left[\{2\,(\lambda+2\mu)+\lambda\tau^2r^2\}\,\beta^2+(\lambda+2\mu)\left(\beta'^2+\frac{2\beta\beta'}{r}\right)r^2+2\mu\int\beta'^2r\,dr\right]=0.$$

Similarly, the second body-stress equation,

$$\frac{\partial\widehat{xy}}{\partial x}+\frac{\partial\widehat{yy}}{\partial y}+\frac{\partial\widehat{yz}}{\partial z}=0,$$

gives

$$\frac{\partial}{\partial y}\left[\{2(\lambda+2\mu)+\lambda\tau^2r^2\}\beta^2+(\lambda+2\mu)\left(\beta'^2+\frac{2\beta\beta'}{r}\right)r^2+2\mu\int\beta'^2r\,dr\right]=0.$$

The third is obviously identically satisfied.

21—*Differential Equation Satisfied by* β

Hence β which is a function of r alone can be determined from the differential equation

$$[2(\lambda+2\mu)+\lambda\tau^2r^2]\beta^2+(\lambda+2\mu)\left(\beta'^2+\frac{2\beta\beta'}{r}\right)r^2+2\mu\int r\beta'^2\,dr=\text{K},$$

where K is a constant of integration.

If we put

$$\eta=\frac{\lambda}{2(\lambda+\mu)},\quad\text{and}\quad c=\frac{1-2\eta}{1-\eta},$$

the above equation becomes

$$[2+(1-c)\tau^2r^2]\beta^2+\left(\beta'^2+\frac{2\beta\beta'}{r}\right)r^2+c\int r\beta'^2\,dr=\text{K}_1,$$

where K_1 is also a constant.

Let us again put

$$(1-c)^{\frac{1}{2}}\tau r=\text{S},$$

and we get

$$(1+\text{S}^2)\beta^2+(\text{S}\beta'+\beta)^2+c\int\text{S}\beta'^2\,ds=\text{K}_1,\quad\text{where}\quad\beta'=\frac{d\beta}{d\text{S}}. \quad . \quad . \qquad (33)$$

It may be remarked that since $\frac{1}{2}>\eta>0$, we have $1>c>0$.

22—*The* β*-Series*

If we put $\text{S}^2=t$ in (33) we get

$$(1+t)\beta^2+\left(2t\frac{d\beta}{dt}+\beta\right)^2+2c\int t\left(\frac{d\beta}{dt}\right)^2dt=\text{K}_1. \quad . \quad . \quad . \quad . \qquad (34)$$

We shall try a solution in series of the form

$$\beta=\text{A}_0+\text{A}_1t+\text{A}_2t^2+\text{A}_3t^3+\ldots\text{A}_nt^n+\ldots.$$

Substituting this value in (34), we get

$$(1+t)(\text{A}_0+\text{A}_1t+\text{A}_2t^2+\ldots\text{A}_nt^n+\ldots)^2+(\text{A}_0+3\text{A}_1t+5\text{A}_2t^2+\ldots(2n+1)\text{A}_nt^n+\ldots)^2$$
$$+2c\int t(\text{A}_1+2\text{A}_2t+3\text{A}_3t^2+\ldots n\text{A}_nt^{n-1}+\ldots)^2\,dt=\text{K}_1.$$

Comparing the coefficients we get

$$\begin{aligned} 2A_0^2 &= K_1, \\ A_1 &= -\tfrac{1}{8}A_0, \\ A_2 &= \frac{A_0\,(6-c)}{12\,.\,8^2}, \\ A_3 &= -A_0\frac{c^2}{9\,.\,8^4}, \end{aligned} \qquad (35)$$

and so on.

Though the numerical value of A_3/A_0 is less than 0·00003 the region of convergency of the infinite series remains unknown. We shall now prove that the series at least converges for all values of $t \leq 1$.

23—*Convergency of the β-Series*

Differentiating (34) with respect to t, we get

$$\beta^2 + 2\beta\beta'(1+t) + 2(2t\beta' + \beta)(2t\beta'' + 3\beta') + 2ct\beta'^2 = 0,$$

where

$$\beta' = \frac{d\beta}{dt}, \quad \text{and} \quad \beta'' = \frac{d^2\beta}{dt^2}.$$

Dividing by β^2 we have

$$1 + 2(1+t)\frac{\beta'}{\beta} + 2\left(2t\frac{\beta'}{\beta} + 1\right)\left(2t\frac{\beta''}{\beta} + 3\frac{\beta'}{\beta}\right) + 2ct\frac{\beta'^2}{\beta^2} = 0.$$

We now put $\gamma = \beta'/\beta$, so that $\gamma' = d\gamma/dt$ is given by $\gamma' + \gamma^2 = \beta''/\beta$. Our last equation now becomes

$$4t\gamma'(2t\gamma+1) + 4t\gamma^2(2t\gamma+1) + 6\gamma(2t\gamma+1) + 2ct\gamma^2 + (2t\gamma+1) + 2\gamma = 0.$$

If $c = 2$ we get

$$(2t\gamma + 1)(4t\gamma' + 4t\gamma^2 + 8\gamma + 1) = 0,$$

and hence either

$$2t\gamma + 1 = 0,$$

or

$$4t\gamma' + 4t\gamma^2 + 8\gamma + 1 = 0.$$

$2t\gamma + 1 = 0$ gives $\beta = K_2 t^{-\frac{1}{2}}$, where K_2 is a constant. It can be easily seen that this value of β satisfies (34) if $c = 2$.

By means of the successive substitutions $t\gamma = \lambda = uv$, $v = 1/t$, we can reduce

$$4t\gamma' + 4t\gamma^2 + 8\gamma + 1 = 0$$

to a RICCATI's equation which can be solved in terms of BESSEL's functions. But we have already seen that $c \not> 1$, and hence for our purpose this solution of $c = 2$ is of no use.

Let us now revert to our equation in γ, and put $2t\gamma + 1 = y$, so that $2t\gamma' + 2\gamma = y'$. It now reduces to

$$4tyy' + 2ty + (y-1)(2y^2 + cy + 2 - c) = 0. \quad \dots \dots \quad (36)$$

If we put $T = t + 2(y-1)$ and $y - 1 = Y$ in (36), we get

$$4(1+Y)(T-2Y)\left(\frac{dT}{dY} - 2\right)[2(1+Y)(T-2Y) + Y\{2(1+Y)^2 + c(1+Y) + 2 - c\}] = 0,$$

or

$$\frac{dT}{dY} = \frac{2Y[2Y^2 + (4+c)Y + 4]}{2T(1+Y) + Y^2(2Y+c)}.$$

The orthogonal trajectory of the family of curves, say, $T = f(Y)$, given by the above equation is

$$\frac{dT}{dY} + \frac{T(1+Y)}{Y[2Y^2 + (4+c)Y + 4]} + \frac{Y^2(2Y+c)}{2Y[Y^2 + (4+c)Y + 4]} = 0,$$

which is a linear equation of the first order in T. Its solution is

$$Te^{\int\frac{(1+Y)\,dY}{Y[2Y^2+(4+c)Y+4]}} = c_1 - \int e^{\int\frac{(1+Y)\,dY}{Y[2Y^2+(4+c)Y+4]}} \times \frac{Y^2(2Y+c)\,dY}{2Y[2Y^2 + (4+c)Y + 4)},$$

or

$$TI_1 = c_1 + 2I_1Y - \tfrac{5}{2}\int I_1\,dY, \quad \dots \dots \dots \quad (37)$$

where c_1 is a constant of integration and $I_1 = e^{\int\frac{(1+Y)\,dY}{Y[2Y^2+(4+c)Y+4]}}$.

Now

$$\int\frac{2(1+Y)\,dY}{Y[2Y^2 + (4+c)Y + 4]} = \int\left[\frac{1}{2Y} - \frac{2Y + 2 + \frac{1}{2}c - (2 - \frac{1}{2}c)}{4[Y^2 + (2 + \frac{1}{2}c)Y + 2]}\right]dY$$

$$= \tfrac{1}{2}\log\frac{Y}{[Y^2 + (2+\frac{1}{2}c)Y + 2]^{\frac{1}{2}}} + \frac{1}{32}\cdot\frac{4-c}{(16 - 8c - c^2)}\tan^{-1}\frac{4(1+Y)+c}{(16 - 8c - c^2)^{\frac{1}{2}}},$$

and hence we can write down the value of I_1.

If we could easily represent graphically the system of curves given by (37) their trajectories would give the solution of the differential equation in y. Since this is not particularly easy we revert to our equation in y and put $y = 1/z$. (36) now gives

$$4t\frac{dz}{dt} - 2tz^2 - 2 + (2-c)z - 2(1-c)z^2 + (2-c)z^3 = 0.$$

Again let $Z = V + d$, where d is a constant as yet undetermined. The above equation now reduces to

$$4t\frac{dV}{dt} - 2t(V+d)^2 - 2 + (2-c)(V+d) - 2(1-c)(V+d)^2 + (2-c)(V+d)^3 = 0.$$

Putting the constant term equal to zero, we have

$$-2+(2-c)\,d-2\,(1-c)\,d^2+(2-c)\,d^3=0,$$

or

$$(d-1)\,[(2-c)\,d^2+cd+2]=0.$$

Therefore, either $d=1$, or $(2-c)\,d^2+cd+2=0$, the roots of which are

$$d=\frac{-c\pm(c^2+8c-16)^{\frac{1}{2}}}{2\,(2-c)},$$

and since $c<1$, both these values of d are imaginary. Hence the constant term can be put equal to zero by only one real value of d, *i.e.*, 1. This remark is important.

For $d=1$ we have

$$t\frac{dV}{dt}+V=\tfrac{1}{2}t+tV-\tfrac{1}{4}\,(4-c)\,V^2+\tfrac{1}{2}tV^2-\tfrac{1}{4}\,(2-c)\,V^3. \quad . \quad . \qquad (38)$$

This type of equation has been studied by Briot and Bouquet.* They have proved that it possesses a regular integral that vanishes with t, and that that is the only integral vanishing with t that is possessed by the equation. Poincaré† has further proved that an equation of the above type cannot have any non-regular integral vanishing with t.

Let the infinite series which satisfies (38) be given by

$$V=c_1t+c_2t^2+\ldots c_nt^n+\ldots.$$

The first few coefficients are

$$c_1=\tfrac{1}{4},$$

$$c_2=\frac{1}{16}\left(1+\frac{1}{12}c\right),$$

$$c_3=\frac{7}{8^3}+\frac{11c}{3\cdot8^3}+\frac{c^2}{12\cdot8^3}$$

$$=\frac{1}{8^2}\,(1+\tfrac{1}{2}c)\left(\frac{21}{24}+\frac{1}{48}c\right),$$

$$c_4=\frac{13}{10\cdot8^3}+\frac{29c}{3\cdot8^4}+\frac{5c^2}{36\cdot8^3}+\frac{c^3}{45\cdot8^4},$$

$$c_5=\frac{179}{120\cdot16^3}+\frac{487c}{15\cdot16^4}+\frac{137c^2}{18\cdot16^4}+\frac{271c^3}{540\cdot16^4}+\frac{c^4}{135\cdot16^4},\ \text{etc.}$$

We have still to find a radius of convergence of the V-series.

* 'J. Éc. Polyt. Paris,' vol. 21, p. 172 (1856).

† *Ibid.*, vol. 28, p. 13 (1878).

The general coefficient c_n is given by

$$(n+1)\, c_n = c_{n-1} + \tfrac{1}{2} \text{ coefficient of } t^{n-1} \text{ in } V^2$$
$$- \tfrac{1}{4}(4-c) \text{ coefficient of } t^n \text{ in } V^2$$
$$- \tfrac{1}{4}(2-c) \text{ coefficient of } t^n \text{ in } V^3.$$

It need hardly be pointed out that to determine the coefficient of t^n in V^2 or V^3 only terms up to $c_{n-1}\, t^{n-1}$ in V are required. Now consider the equation

$$2V_1 = \tfrac{1}{2}t + tV_1 + \tfrac{1}{2}t\, V_1^2.$$

If an infinite series

$$V_1 = c'_1 t + c'_2 t^2 + c'_3 t^3 + \ldots c'_n t^n + \ldots$$

satisfies the above equations we have

$$c'_1 = \tfrac{1}{4},$$
$$c'_2 = \tfrac{1}{8},$$
$$c'_3 = \frac{5}{64}, \text{ and so on.}$$

It is obvious that c_1, c_2, c_3 and c'_1, c'_2, c'_3 are all positive and that $c'_1 = c_1$, $c'_2 > c_2$, $c'_3 > c_3$. We assume that both these laws hold good from c_2 and c'_2 up to c_{n-1} and c'_{n-1}. We now prove that it is true for c_n and c'_n.

c'_n is given by

$$c'_n = c'_{n-1} + \tfrac{1}{2} \text{ coefficient of } t^{n-1} \text{ in } V_1^2.$$

Since for the coefficient of t^{n-1} in V_1^2 terms up to $c'_{n-2}\, t^{n-2}$ need only be taken and since all c''s up to c'_{n-1} have been supposed to be positive and greater than the corresponding c's which have also been assumed to be positive up to c_{n-1} we have on comparing the general values of c_n and c'_n

$$c'_n > c_n.$$

Now V_1 is given by

$$tV_1^2 + 2V_1(t-2) + t = 0.$$

Therefore

$$V_1 = \frac{-(t-2) \pm 2(1-t)^{\frac{1}{2}}}{t}.$$

Obviously to get the infinite series,

$$V_1 = c'_1 t + c'_2 t^2 + \ldots c'_n t^n + \ldots,$$

we must take the minus sign. Thus the series converges for all values of $t \leq 1$. And since after the coefficient of the first term in V we have $c_n < c'_n$ the V-series also converges for all values of $t \leq 1$. For $t = 1$, $V_1 = 1$, and it is obviously less than 1 if $t < 1$. Hence, V is also less than 1 if $t \leq 1$.

Thus we have proved that the infinite series,

$$V = c_1 t + c_2 t^2 + c_3 t^3 + \dots c_n t^n + \dots,$$

is absolutely and uniformly convergent for all values of $t \leq 1$, and that in this region of convergency V is always less than 1.

Now

$$V = z - 1 = \frac{1}{y} - 1 = \frac{1}{2t\gamma + 1} - 1 = -\frac{2t\gamma}{2t\gamma + 1}.$$

Therefore

$$2t\gamma = -\frac{V}{1 + V}.$$

But since

$$\beta' = \gamma\beta$$

$$\beta = Ae^{\int \gamma\, dt}$$

$$= Ae^{-\frac{1}{2}\int \frac{V}{1+V}\cdot\frac{dt}{t}},$$

where A is a constant.

Again,

$$\frac{V}{1+V} = V(1 - V + V^2 - V^3 + \dots), \quad \dots\dots \quad (39)$$

if $V < 1$ or $t \leq 1$.

Since V is absolutely convergent for $t \leq 1$, V^2, V^3, V^4, V^5, etc., are all absolutely convergent series in the same domain. Hence we can rearrange the terms on the right-hand side of (39) and get

$$\frac{V}{1+V} = a_1 t + a_2 t^2 + \dots a_n t^n + \dots$$

$$= \text{an absolutely convergent series}$$

$$= -2\,Ut \text{ (say)}.$$

Therefore

$$\beta = Ae^{\int U\,dt} = Ae^{W} \text{ (say)},$$

where W is also an absolutely convergent series.

Therefore

$$\beta = A\left[1 + W + \frac{W^2}{2!} + \frac{W^3}{3!} + \dots\right],$$

and since W, W^2, W^3, etc., are all absolutely convergent series we can once again rearrange the terms and get

$$\beta = A_0 + A_1 t + A_2 t^2 + \dots A_n t^n + \dots,$$

a series which in turn should be absolutely convergent for $t \leq 1$. To calculate the A's it is obviously better to substitute the β-series in (34) and then compare the coefficients of the various powers. This we have already done in (35). Hence we can write

$$\beta = A_0 \left[1 - \tfrac{1}{8}t + \frac{6-c}{12 \,.\, 8^2}\, t^2 - \frac{c^2}{9 \,.\, 8^4}\, t^3 + \dots\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It should be noticed that $t \leq 1$ is only a lower limit to the radius of convergence of the power series given by (40). Actually this radius may be much greater than unity.

From physical considerations we see that the exact solution of the differential equation (34), which holds good for $t = 0$ should be valid for all magnitudes of the twist. Hence, it cannot have any singularities for all real and positive values of t. The value of β for $t > 1$ may, therefore, be obtained by analytic continuation, taking new initial values.

In what follows we shall suppose that $t \leq 1$ unless otherwise stated.

23—*Satisfaction of the Boundary Conditions*

Now for satisfying the boundary conditions it is convenient to put $\beta = A_0 F(r)$, where $F(r)$ is a known function of the radius vector r, and A_0 is a constant whose value is still to be determined from the boundary conditions.

The first two boundary conditions

$$\widehat{xx} \cos(xn) + \widehat{xy} \cos(yn) = 0,$$

and

$$\widehat{xy} \cos(xn) + \widehat{yy} \cos(yn) = 0,$$

give

$$\lambda \left[(1-\beta^2) - \tfrac{1}{2}r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) + \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2\right]$$
$$+ \mu(1-\beta^2) - \mu r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) = 0, \quad (41)$$

which is to hold good over $r = a$, where a is the radius of the circular cylinder after strain. If a_0 be its radius before strain we have the relation

$$a_0{}^2 = a^2\beta_0{}^2, \qquad \text{where} \qquad \beta_0 = A_0 F(a). \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The third boundary condition,

$$\widehat{xz} \cos(xn) + \widehat{yz} \cos(yn) = 0,$$

is obviously identically satisfied.

We can now rewrite (41) as

$$(\lambda + \mu) - \lambda\,(\alpha - \tfrac{1}{2}\alpha^2) = \tfrac{1}{2}A_0^2\,[(\lambda + 2\mu)\,\{aF'(a) + F(a)\}^2 + \lambda\,(1 + \tau^2 a^2)\,\{F(a)\}^2],$$

or

$$(3\lambda + 2\mu) - \lambda\,(1 - \alpha)^2 = A_0^2\,[(\lambda + 2\mu)\,\{aF'(a) + F(a)\}^2 + \lambda\,(1 + \tau^2 a^2)\,\{F(a)\}^2], \quad \ldots\ldots \quad (43)$$

where, of course, $F'(a) = \left[\dfrac{dF(r)}{dr}\right]_{r=a}$.

The tractions on any cross-section are, of course, statically equivalent to a single couple whose axis is the axis of z. We have to show that

$$\iint \widehat{xz}\,dx\,dy = 0, \qquad \iint \widehat{yz}\,dx\,dy = 0, \qquad \iint \widehat{zz}\,dx\,dy = 0, \quad .\ . \quad (44.1)$$

$$\iint y\widehat{zz}\,dx\,dy = 0, \qquad \iint - x\widehat{zz}\,dx\,dy = 0. \quad \ldots\ldots\ldots \quad (44.2)$$

Now from (32) we have $\widehat{xz} = -\mu\tau\beta^2 y$, $\widehat{yz} = \mu\tau\beta^2 x$; also $\widehat{zz}$ and β^2 are both functions of r only. Since the axis of z passes through the centre of the circular cross-sections, we can see that all the above equations are satisfied excepting the third in (44.1) which requires

$$\int_0^a 2\pi r\widehat{zz}\,dr = 0,$$

or

$$[\lambda + (\lambda + 2\mu)\,(\alpha - \tfrac{1}{2}\alpha^2)]\,a^2 - \tfrac{1}{2}A_0^2\lambda \int_0^a \{rF'(r) + F(r)\}^2\,r\,dr$$
$$- \tfrac{1}{2}A_0^2 \int_0^a \{\lambda + (\lambda + 2\mu)\,\tau^2 r^2\}\,\{F(r)\}^2\,r\,dr = 0,$$

or

$$(3\lambda + 2\mu) - (\lambda + 2\mu)\,(1 - \alpha)^2$$
$$= \frac{2A_0^2}{a^2} \int_0^a [\lambda\,\{r\,F'(r) + F(r)\}^2 + \{\lambda + (\lambda + 2\mu)\,\tau^2 r^2\}\,\{F(r)\}^2]\,r\,dr. \quad (45)$$

We see that (43) and (45) are both necessary and sufficient to determine the unknown constants A_0^2 and $(1 - \alpha)^2$. The value of the torsional couple N has been obtained in § 25 *infra*.

Since all the boundary conditions and the body-stress equations have been satisfied, the analytical solution of the problem is complete.

We notice that (43) and (45) determine A_0^2 and $(1 - \alpha)^2$, and not A_0 and $(1 - \alpha)$. Since $w = z - z'$ we have $z' = (1 - \alpha)\,z$. Now z is positive or negative according as z' is positive or negative. Hence we should take the positive sign with the square root of the value of $(1-\alpha)$. In like manner, since $r' = r\beta = A_0 r\,F(r)$, we should again take the positive sign with A_0.

As far as the stresses go both A_0 and $(1-\alpha)$ appear in the second degree in them. It is in the displacements that one has to be careful regarding the sign of A_0 and $(1-\alpha)$.

24—*Determination of* α *and* A_0

We shall now calculate expressions for the values of α and A_0^2 by using the infinite series,

$$\beta = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots A_n t^n + \dots,$$

which is at least absolutely convergent for $t \leq 1$.

We have

$$\begin{aligned}\beta^2 &= A_0^2\,\{F(r)\}^2\\ &= A_0^2 + 2A_0A_1 t + (A_1^2 + 2A_0A_2)\,t^2 + (2A_0A_3 + 2A_1A_2)\,t^3 + \dots,\end{aligned} \quad (46)$$

and

$$\begin{aligned}\left[\frac{d}{dr}(r\beta)\right]^2 &= A_0^2\left[\frac{d}{dr}\,rF(r)\right]^2\\ &= \left[\frac{dt}{dr}\cdot\frac{d}{dt}\cdot\frac{1}{\tau(1-c)^{\frac{1}{2}}}\{A_0t^{\frac{1}{2}} + A_1t^{\frac{3}{2}} + A_2t^{\frac{5}{2}} + A_3t^{\frac{7}{2}} + \dots\}\right]^2\\ &= (A_0 + 3A_1t + 5A_2t^2 + 7A_3t^3 + \dots)^2\\ &= A_0^2 + 6A_0A_1t + (9A_1^2 + 10A_0A_2)\,t^2 + 2\,(7A_0A_3 + 15A_1A_2)\,t^3 + \dots,\end{aligned} \quad (47)$$

since $t = (1-c)\,\tau^2 r^2$.

After dividing both sides of (43) by $(\lambda + 2\mu)$ it becomes

$$(3-2c) - (1-c)\,(1-\alpha)^2 = A_0^2\left[\frac{d}{dr}\{rF(r)\}\right]^2_{r=a} + A_0^2\,(1-c)\,(1+\tau^2a^2)\,\{F(a)\}^2, \quad (48)$$

and in like manner we can write (45) as

$$(3-2c) - (1-\alpha)^2 = \frac{2A_0^2}{a^2}\int_0^a\left[(1-c)\left\{\frac{d}{dr}\,rF(r)\right\}^2 + \{(1-c)+\tau^2r^2\}\{F(r)\}^2\right]r\,dr. \quad (49)$$

Substituting the values of $[F(r)]^2$ and $\left[\frac{d}{dr}\,rF(r)\right]^2$ from (46) and (47) in (48) and (49) we have if $t_0 = (1-c)\,\tau^2a^2$

$$\begin{aligned}(3-2c) - (1-c)\,(1-\alpha)^2 &= A_0^2\,(2-c) + t_0\,[A_0^2 + 2A_0A_1\,(4-c)]\\ &\quad + t_0^2\,[2A_0A_1 + 2\,(5A_1^2 + 6A_0A_2) - c\,(A_1^2 + 2A_0A_2)]\\ &\quad + t_0^3\,[A_1^2 + 2A_0A_2 + 16\,(A_0A_3 + 2A_1A_2) - 2c\,(A_0A_3 + A_1A_2)] + \dots,\end{aligned} \quad (50)$$

and

$$(3-2c)(1-c)-(1-c)(1-\alpha)^2 = (1-c)^2\,[2A_0^2+4A_0A_1t_0+\tfrac{2}{3}(5A_1^2+6A_0A_2)\,t_0^2+4(A_0A_3+2A_1A_2)] + [\tfrac{1}{2}A_0^2t_0+\tfrac{2}{3}A_0A_1t_0^2+\tfrac{1}{4}(A_1^2+2A_0A_2)\,t_0^3+\dots]. \quad (51)$$

On subtraction (50) and (51) give

$$\begin{aligned} c(3-2c) = {} & cA_0^2(3-2c)+t_0\,[2A_0A_1(2+3c-2c^2)+\tfrac{1}{2}A_0^2] \\ & + t_0^2\,[\tfrac{4}{3}A_0A_1+2(5A_1^2+6A_0A_2)-c(A_1^2+2A_0A_2) \\ & \qquad -\tfrac{2}{3}(1-c)^2(5A_1^2+6A_0A_2)] \\ & + t_0^3\,[\tfrac{3}{4}A_1^2+\tfrac{3}{2}A_0A_2+16(A_0A_3+2A_1A_2)-2c(A_0A_3+A_1A_2) \\ & \qquad -4(1-c)^2(A_0A_3+2A_1A_2)]+\dots . \end{aligned}$$

Putting the values of A_1, A_2, etc., and dividing both sides by $c(3-2c)$ we get

$$1 = A_0^2\left[1-\tfrac{1}{4}t_0+\tfrac{1}{24}(1-\tfrac{1}{16}c)\,t_0^2-\frac{24-20c+3c^2}{4\,.\,8^3\,(3-2c)}\,t_0^3+\dots\right], \quad . \quad (52)$$

which gives the value of A_0^2.

Since A_0^2 is to be real, the numerical value of

$$[\tfrac{1}{4}t_0-\tfrac{1}{24}(1-\tfrac{1}{16}c)\,t_0^2]+\frac{24-20c+3c^2}{4\,.\,8^3\,(3-2c)}\,t_0^3\dots$$

is less than 1, and hence we can get A_0^2 as an infinite power series in t_0 given by

$$A_0^2 = 1+\tfrac{1}{4}t_0+\tfrac{1}{48}(1+\tfrac{1}{8}c)\,t_0^2+\left[-\frac{1}{12\,.\,8^2}+\frac{c}{3\,.\,8^3}+\frac{c^2}{12\,.\,8^3\,(3-2c)}\right]t_0^3+\dots . \quad . \; . \; . \; . \; . \; . \; . \quad (53)$$

For α we substitute the values of A_0, A_1, A_2, etc., in (51) and get

$$(1-c)\,[(3-2c)-(1-\alpha)^2] = A_0^2\left[2(1-c)^2+\tfrac{1}{2}c(2-c)\,t_0 -\frac{c}{3\,.\,8^2}(33-18c+c^2)\,t_0^2+\frac{c}{6\,.\,8^3}(50-\tfrac{97}{3}c+\tfrac{14}{3}c^2-\tfrac{1}{3}c^3)\,t_0^3\dots\right].$$

Putting the value of A_0^2 from (53) we have

$$(1-c)\,[(3-2c)-(1-\alpha)^2] = 2(1-c)^2+\tfrac{1}{2}t_0+\tfrac{1}{24}t_0^2+\dots,$$

or

$$(1-c)(1-\alpha)^2 = (1-c)-\tfrac{1}{2}t_0-\tfrac{1}{24}t_0^2, \quad . \; . \; . \; . \; . \; . \; . \quad (54)$$

which gives the value of $(1-\alpha)^2$.

As we have already pointed out we should take both A_0 and $(1-\alpha)$ with a positive sign. (52) now gives

$$A_0 = 1 + \tfrac{1}{8}t_0 + \tfrac{1}{384}(1+c)\,t_0^2 + \dots, \qquad (55)$$

and (54)

$$(1-\alpha)^2 = 1 - \frac{t_0}{2(1-c)} - \frac{t_0^2}{24(1-c)}\dots$$

$$= 1 - \tfrac{1}{2}\tau^2a^2 - \tfrac{1}{24}(1-c)\,\tau^4a^4\dots,$$

i.e.,

$$\alpha = \tfrac{1}{4}\tau^2a^2 + \tfrac{1}{96}(5-2c)\,\tau^4a^4 + \dots. \qquad (56)$$

For future use we can now write down the value of β^2 as

$$\beta^2 = 1 - \tfrac{1}{4}(1-c)\,\tau^2(r^2-a^2) + \frac{1}{6.8^2}(1-c)^2\,\tau^4\,[r^4(12-c) - 24a^2r^2 + (8+c)\,a^4]. \qquad (57)$$

It is obvious that to the first power of τa, $A_0 = 1$, $\beta = 1$, and $\alpha = 0$. In such a case we fall back on SAINT-VENANT's solution of the Torsion problem.

25—*Torsional Couple*

The twisting couple N per unit length is given by

$$N = \mu\tau\int_0^a \beta^2r^2\,.\,2\pi r\,dr.$$

Substituting the value of β^2 from (57) we have

$$N = \tfrac{1}{2}\pi\mu\tau a^4\left[1 + \tfrac{1}{12}(1-c)\,\tau^2a^2 - \frac{(1-c)^2}{12.16}(1-\tfrac{1}{4}c)\,\tau^4a^4\dots\right]. \qquad (58)$$

If τa is so small that all its powers beyond the first can be neglected, we get

$$N = \tfrac{1}{2}\pi\mu\tau a^4 = \mu\tau I,$$

where I is the moment of inertia of the cross-section round the axis of the cylinder. This result is already well known. If only powers of τa beyond the second can be neglected we see that $N = \mu\tau I$ remains unchanged, a result obtained by SAINT-VENANT and also stated without proof by YOUNG.* If τ^3a^3 can be retained the term to be added to $\mu\tau I$ to get the next approximation is

$$\tfrac{1}{24}\pi\mu(1-c)\,\tau^3a^6 = \tfrac{1}{24}\pi\,.\,\frac{\mu\eta}{1-\eta}\,\tau^3a^6. \qquad (59)$$

* "Lectures on Natural Philosophy," vol. I, p. 139.

Saint-Venant in a footnote given on pp. 477-8 of his memoir on Torsion calculates the correction for N on the supposition that longitudinal fibres suffer a stretch. He obtains the correction as

$$\tfrac{1}{24}\pi E\tau^3 a^6 = \tfrac{1}{24}\pi \,.\, 2\mu\,(1+\eta)\,\tau^3 a^6.$$

The difference between his correction and ours is

$$\tfrac{1}{24}\pi\mu \,.\, \frac{2-\eta-2\eta^2}{1-\eta} \,.\, \tau^3 a^6.$$

This can only be zero if

$$2\eta^2+\eta-2=0$$

or

$$\eta = \tfrac{1}{4}\,(\sqrt{17}-1),$$

which is obviously greater than $\frac{1}{2}$. Hence the difference can never vanish.

It appears that if the longitudinal stress modulus E be replaced by the constant $\mu\,(1-c)$ the two results become the same.

I may here point out another inference from the above result. Balinkin* compares the observed values of the axial stresses in long rectangular plates under torsion with the calculated values based on an analysis similar to Saint-Venant's and taken from Weber.† If the results of Saint-Venant's approximate theory do not agree with our theory in such a simple case as that of a circular cylinder it is obvious that we cannot expect a very good agreement between the two values of the axial stresses in a complicated case like that of a rectangular plate. In his experiments Balinkin twists the plate through an angle of 60° which no one can call a small angle of twist.

26—*Comparison Table*

Let

$N_a = \frac{1}{2}\pi\mu\tau a^4$

= the torsional couple given by the ordinary theory, and N_e = the same couple given by our more exact theory from (58).

Since (57) shows that N_e/a^3 is a function of τa we have compared the values of N_a/a^3 and N_e/a^3 for different values of τa, and have calculated the percentage error that arises from taking N_a/a^3 for N_e/a^3.

We take $\eta = \frac{1}{4}$ or $c = \frac{2}{3}$. The condition of convergency of the β-series gives $\tau a \leq \sqrt{3}$.

* 'Phys. Rev.,' vol. 30, p. 520 (1927).

† 'Forsch. Arb. vers. deut. Ing.,' No. 249.

TABLE I

τa.	($N_a/a^3 = \mu K_1$) K_1	($N_e/a^3 = \mu K_2$) K_2	% error
$\frac{1}{12}\pi$	0·4112	0·4120	0·19
$\frac{1}{6}\pi$	0·8235	0·8288	0·64
$\frac{1}{4}\pi$	1·2337	1·2548	1·68
$\frac{1}{3}\pi$	1·6449	1·6951	2·96
$\frac{1}{2}\pi$	2·4674	2·6364	6·41

From the above table we see that the percentage error increases with τa. When $\tau a = \frac{1}{2}\pi$ this error is as much as 6·41%.

27—*Lines of Principal Stress*

If

$$l = \frac{x}{r}, \qquad m = \frac{y}{r}, \qquad n = 0$$

$$l' = -\frac{y}{r}, \qquad m' = \frac{x}{r}, \qquad n' = 0,$$

we have

$$\widehat{rr} = l^2\widehat{xx} + m^2\widehat{yy} + 2lmxy$$

$$= \lambda\left[(1-\beta^2) + \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\left(\beta^2\tau^2 + \beta'^2 + \frac{2\beta\beta'}{r}\right)r^2\right] + \mu(1-\beta^2) - \mu\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)r^2, \quad (60.1)$$

$$\widehat{r\theta} = ll'\widehat{xx} + mm'\widehat{yy} + (lm' + l'm)\widehat{xy}$$

$$= 0, \quad (60.2)$$

$$\widehat{\theta\theta} = l'^2\widehat{xx} + m'^2\widehat{yy} + 2l'm'\widehat{xy}$$

$$= \lambda\left[(1-\beta^2) + \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\left(\beta^2\tau^2 + \beta'^2 + \frac{2\beta\beta'}{r}\right)r^2\right] + \mu(1-\beta^2), \quad (60.3)$$

$$\widehat{rz} = 0, \quad (60.4)$$

$$\widehat{\theta z} = \mu\tau\beta^2 r, \quad (60.5)$$

and $\widehat{zz}$ is given by (32.3). In (60.1) β' stands for $d\beta/dr$.

Thus the principal stresses at any point on the z-axis are $\widehat{rr}$, $\widehat{\theta\theta}$ and $\widehat{zz}$. But this is not so at any other point of a cross-section. $\widehat{rr}$ still remains the principal stress, but $\widehat{\theta\theta}$ and $\widehat{zz}$ do not. If P and Q be the principal stresses in the (θ, z) plane, and

ϕ, $\frac{1}{2}\pi + \phi$ be the angles they make with the direction of θ, we have, by a well-known result,

$$\tan 2\phi = \frac{2\widehat{\theta z}}{\widehat{\theta\theta} - \widehat{zz}},$$

$$\mathrm{P} - \mathrm{Q} = [(\widehat{\theta\theta} - \widehat{zz})^2 + 4\widehat{\theta z}^2]^{\frac{1}{2}} = + \mathrm{R} \text{ (say)},$$

$$\mathrm{P} + \mathrm{Q} = \widehat{\theta\theta} + \widehat{zz},$$

the ambiguity of ϕ being settled by

$$\widehat{\theta\theta} - \widehat{zz} = \mathrm{R} \cos 2\phi,$$

$$2\widehat{\theta z} = \mathrm{R} \sin 2\phi.$$

From (60.3) and (32.3) we have

$$\widehat{\theta\theta} - \widehat{zz} = \mu (1 - \alpha)^2 - \beta^2 (1 - \tau^2 r^2).$$

Therefore

$$\tan 2\phi = \frac{2\tau\beta^2 r}{(1 - \alpha)^2 - \beta^2 (1 - \tau^2 r^2)}.$$

In the ordinary theory we neglect $\tau^2 a^2$ and all its higher powers, and hence it is that the principal axes at any point of a cross-section are included at an angle of 45° to the axis of the prism. In the present case, it happens only when r satisfies the equation

$$\beta^2 (1 - \tau^2 r^2) = (1 - \alpha)^2,$$

which, if we neglect powers of τa higher than the second, gives

$$r = a \left(\frac{3 - c}{5 - c}\right)^{\frac{1}{2}}.$$

On the surface of the cylinder we get

$$\tan 2\phi = \frac{4}{\tau a} \text{ (approx.)},$$

or

$$\phi = \tfrac{1}{4}\pi - \tfrac{1}{8}\tau a.$$

28—*Value* of $\widehat{zz}$

The value of the axial stress $\widehat{zz}$ can be obtained with the help of (32.3), (56) and (57). A little calculation gives

$$\widehat{zz} = \tfrac{1}{2} (\lambda + 2\mu)\, c\, (2 - c)\, \tau^2 (\tfrac{1}{2}a^2 - r^2),$$

neglecting $\tau^3 a^3$, $\tau^4 a^4$, etc. This vanishes when

$$r = a/\sqrt{2}.$$

If a_0 is the radius of the cylinder before strain, we have

$$a_0 = a\,(\beta)_{r=a},$$

or using (57)

$$a_0 = a\left\{1 - \frac{1}{3\,.\,8^2}(1-c)^2\,\tau^4 a^4\,\ldots\right\},$$

which shows that a_0 remains unchanged to the fourth power of τa. Hence a can always be replaced by a_0 in the foregoing approximations.

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